THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 8 Problems
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- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Let $(R, +, \times)$ be a ring, and $R^\times$ be the set of units in R, show that $(R^\times, \times)$ is a group. Find the groups of units for the following rings:
   (a) $\mathbb{Z}_{20}$.
   (b) The ring of n by n matrices with complex coefficients $M_{n\times n}(\mathbb{C})$.
   (c) The polynomial ring $R[x]$ where R is an integral domain.
2. Let G be an abelian groups, denote $R = \operatorname{End}(G)$ the set of group homomorphism from G to itself, define operations $+, \circ$ on R by $(\varphi + \psi)(g) := \varphi(g) \cdot \psi(g)$ where $\cdot$ is the group operation on G and $(\varphi \circ \psi)(g) = \varphi(\psi(g))$. Prove that $\operatorname{End}(G)$ is a ring and compute $\operatorname{End}(\mathbb{Z}_p)$ where p is a prime.
3. Let R be a finite commutative ring, show that every nonzero element is either a unit or a zero divisor. What if R is not assumed to be finite?
4. Let R, S be rings, define binary operations $+, *$ on $R \times S$ by $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1) * (r_2, s_2) = (r_1r_2, s_1s_2)$. Prove that $R \times S$ is a ring with these operations. Show that if R, S are non-trivial, then $R \times S$ necessarily contains zero divisors, and hence cannot be an integral domain. $R \times S$ is called the **product ring**.
5. (a) Let S be a non-empty set, and $P(S)$ be the power set, i.e. $P(S) = \{A \subset S\}$ the set of all subsets of S. Define binary operations $+$ and $*$ on $P(S)$ by

   $$A + B := (A \cup B) \setminus (A \cap B) \text{ and } A * B := A \cap B$$

   Prove that $(P(S), +, *)$ is a ring.
   (b) Prove that $A + A = 0$ and $A * A = A$.
   (c) In general, a ring R satisfying $x * x = x$ for all $x \in R$ is called a Boolean ring. Prove that $(\mathbb{Z}/2\mathbb{Z})^n$ is also a Boolean ring. (Here the index n denotes product ring of n copies of $\mathbb{Z}/2\mathbb{Z}$.)
   (d) Prove that a Boolean ring R is always commutative. And that R is an integral domain if and only if $R \cong \mathbb{Z}/2\mathbb{Z}$.